1. Let $c_0 = \{\{\alpha_n\}_{n\geq 1} : \alpha_n \to 0\}$, equipped with the supremum norm. Let $f : c_0 \to \mathbb{C}$ be a linear function such that $f(e_n) = n$. Show that $ker(f)$ is dense in $c_0$.

   **Solution:** Given that $f : c_0 \to \mathbb{C}$ is a linear function such that $f(e_n) = n$ where $e_n$ denotes the sequence whose $n$-th term is 1 and all other terms are zero. We claim that in order to prove that $Ker(f)$ is dense in $c_0$ it suffices to see that $f$ is discontinuous. For, if $f$ is discontinuous, then $f$ is not bounded and hence, for each $n = 1, 2, \cdots$, there exists $x_n \in c_0$ such that $\|x_n\| \leq 1$ and $|f(x_n)| > n$. Thus if $x$ is an element of $c_0$ lying outside $ker(f)$, then $(y_n)$ where $y_n = x - \frac{f(x)}{f(x_n)}x_n$, is a sequence in $ker(f)$ and $\|x - y_n\| = \frac{|f(x)|}{|f(x_n)|}\|x_n\| < \frac{|f(x)|}{n} \to 0$ as $n \to \infty$, showing that $ker(f)$ is dense in $c_0$. Thus we just need to show that $f$ is discontinuous. For each $n$, consider the sequence $x_n = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \frac{1}{n}, 0, 0, 0, \cdots)$, then clearly $x_n \to 0$, $\|x_n\| = 1$ and $|f(x_n)| = n$ so that $f$ is not continuous. □

2. Let H be a complex Hilbert space and let $\Phi : H \to H$ be a linear onto map such that $||\Phi(x)|| = ||x||$ for all $x \in H$. Show that $\Phi$ preserves the inner product.

   **Solution:** Using the polarisation identity we observe that

$$4 < \Phi(x), \Phi(y) >= \sum_{k=0}^{3} \|\Phi(x + i^k y)\|^2$$

   which by the given condition equals $\sum_{k=0}^{3} \|x + i^k y\|^2$ which by another application of the polarisation identity equals $4 < x, y >$. Hence $\Phi$ preserves inner product. □

3. Let $A$ be a commutative complex Banach algebra with identity $e$. Show that $A$ is homomorphic and isometric to a subalgebra of the space of bounded linear operators on a Banach space.

   **Solution:** Let $B(A)$ denote the Banach algebra of bounded linear operators on $A$. Define $\lambda : A \to B(A)$ by $\lambda_a(b) = ab$ for $a, b \in A$ (the notation $\lambda_a$ stands for $\lambda(a)$). Clearly $\lambda$ is unital algebra homomorphism. Now, $\|\lambda_a(b)\| = \|ab\| \leq \|a\|\|b\|$ so that $\|\lambda_a\| \leq \|a\|$. Also we have

$$\|\lambda_a\| = \sup_{\|b\|\leq 1} \|ab\| \geq \|a\|.$$

   Thus $\|\lambda_a\| = \|a\|$ for all $a$ in $A$ and consequently, $A$ is homomorphic and isometric to the subalgebra $\lambda(A)$ of $B(A)$. □

4. Let $M = \{f \in C([0,1]) : f(E) = 0\}$, for some closed set $E \subset [0,1]$. Show that the quotient algebra $C([0,1])/M$ is homomorphic and isometric to $C(E)$.

   **Solution:** Consider the map from $C([0,1]) \to C(E)$ given by $f \to f|_E$(restriction of $f$ to $E$). Clearly this a unital $*$-homomorphism between $C^*$-algebras which, by virtue of the Tietze extension theorem, is surjective also. Kernel of this map is obviously $M$ which is closed also so that the quotient $C([0,1])/M$ is a $C^*$-algebra and the map $C([0,1])/M \to C(E)$ given by $f + M \to f|_E$ is unital $*$-isomorphism and thus is an isometry.

   □

5. Consider the Hilbert space $\ell^2 = \{\{\alpha_n\}_{n\geq 1} : \sum |\alpha_n^2| < \infty\}$. Let $T : \ell^2 \to \ell^2$ be defined by $T(\{\alpha_n\}_{n\geq 1}) = \{\frac{1}{n}\alpha_n\}_{n\geq 1}$. Compute $||T||$, also describe $T^*$.

   **Solution:** We claim that $T^* = T$. To see this, given $\{\alpha_n\}, \{\beta_n\} \in \ell^2$, we note

$$< T\{\alpha_n\}, \{\beta_n\} >= \sum_{n=1}^{\infty} \frac{\alpha_n}{n}\bar{\beta}_n =< \{\alpha_n\}, T\{\beta_n\} > .$$

   Now, $\|T\{\alpha_n\}\| \leq \|\{\alpha_n\}\|$ so that $\|T\| \leq 1$. Now the sequence $e_1$ (whose first element is 1 and all other elements are 0) clearly has norm 1 and $T(e_1) = e_1$ showing that $\|T\| = 1$. □

6. Let $A$ be a commutative Banach algebra with identity $e$. Show that every proper ideal is contained in a maximal ideal.

   **Solution:** Let $A$ be a commutative Banach algebra with identity $e$. Let $I$ be a proper ideal in $A$. Let $X$ be the collection of all proper ideals of $A$ containg $I$ partially ordered by inclusion. Note $X$ is non-empty since $I$ is in $X$. Let $C$ be a chain in $X$. Let $I' = \cup_{I\in C} I$. One can easily see that $I'$ is an ideal of $A$ and that it is a proper ideal so that $I'$ is an upper bound of $C$. Then by Zorn's Lemma $X$ possesses a maximal element, as desired. □

7. Let $X$ be a normed linear space and $Y$ a Banach space. Show that the space of bounded linear operators $\mathcal{L}(X, Y)$ is a Banach space.

   **Solution:** See Rudin Functional Analysis Book, Chapter 4, Theorem 4.1.

8. Let $T : L^1([0,1]) \to L^2([0,1])$ be a bounded linear map. Consider $T^* : L^2([0,1]) \to L^\infty([0,1])$ defined by $T^*(f)(g) = \displaystyle\int T(g)\bar{f}dx$ for $f \in L^2([0,1])$ and $g \in L^1([0,1]$. Show that $T^*$ is a well-defined, bounded linear map.

**Solution:** Using the fact that $L^1([0,1])^* = L^\infty([0,1])$ we note that for any $g \in L^1([0,1]), f \in L^2([0,1]), T(g)\bar{f} \in L^1([0,1])$ and so, $\int T(g)\bar{f}dx$ is a complex number. It is clear that $T^*$ is linear. Now for any $f \in L^2([0,1])$,

$$|T^*(f)(g)| = |\int T(g)\bar{f}dx| \leq \int |T(g)\bar{f}|dx = \|T(g)\bar{f}\|_1 \leq \|T(g)\|_2\|f\|_2 \leq \|T\|\|g\|_1\|f\|_2$$

so that $T^*$ is bounded.

□

9. Let $H$ be a complex Hilbert space. Show that for some infinite discrete set $\Gamma$, there is a bounded linear map from $H$ onto $\ell^2(\Gamma)$.

**Solution:** Let $\{e_i : i \in I\}$ be an orthonormal set in $H$. We know from Bessel's inequality that for any $x$ in $H$,

$$\sum_{i \in I} |< x, e_i >|^2 \leq \|x\|^2 \tag{1}$$

Let us consider the Hilbert space $\ell^2(I)$ consisting of all complex functions $f$ on $I$ such that $\sum_i |f(i)|^2 < \infty$ with the inner product given by

$$< f, g >= \sum_i f(i)\bar{g(i)}$$

for $f, g$ in $\ell^2(I)$. Define a mapping $T$ from $H$ to $\ell^2(I)$ given by $x \in H \to x'$ where $x'(i) =< x, e_i >$. $T$ is obviously linear. It follows from (1) that $T$ is bounded also. Let $P$ be the space of all finite linear combinations of the vectors $e_i$. Note also that $T$ is an isometry of $P$ onto the dense subspace of $\ell^2(I)$ consisting of those functions whose support is a finite subset of $I$. This implies that $T$ is onto.

□

10. Let $H$ be a complex Hilbert space and let $T : H \to H$ be a bounded linear map such that its adjoint $T^* = 0$. Show that $T = 0$.

**Solution:** Note that for any $x, y \in H, < Tx, y >=< x, T^*y >= 0$ which implies that $T = 0$. □